Solutions to Problems in Morton Hamermesh's "Group Theory and its Application to Physical Problems", 2nd Ed. (Dover, 1989)

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Something that bothers me about the undergraduate physics curriculum at Waterloo is that despite the wide array of quantum theory courses you can (and indeed, must) take, students are not required to take a single course in abstract algebra. Even those in mathematical physics needn't so much as look at the pure math calendar to get their degree.

Modern quantum mechanical theory relies very heavily on the language of groups and algebras –terms that only increase in their importance the higher up the chain you go. You hear things like "Poisson bracket" and "Lie group" and perhaps even "Lie Algebra". You hear about Noether's theorem, but only her work on variational calculus. If you take particle physics, you may hear about special unitary groups and that sort of thing.

It hardly suffices to take these terms for granted if one wants to dive into the weeds of modern-day mathematical physics, and there are few places more worthy of building foundational knowledge than in group theory. So, hopefully you try the problems out before looking at my solutions here. That's the only way to really learn, after all. But this will be here should you get truly stuck, and no mathematician is available to consult.

Hamermesh's text does not number the problems, but the sections are broken up nicely enough that they're not hard to find, provided I index those sections here. I will also write out the questions themselves so that searching for them will be even more simple.

CHAPTER 1: ELEMENTS OF GROUP THEORY

1-1 Correspondences and Transformations.

<u>Problem 1-1.1:</u> The cross ratio of four points on a line is defined as

$$\frac{(x_1 - x_2)/(x_3 - x_2)}{(x_1 - x_4)/(x_3 - x_4)} = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)},$$

where x_1, x_2, x_3, x_4 are the coordinates of the four points. Show that the ratio is *invariant* under projective transformation, i.e. that the cross ratio obtained from the image point has the same form as that for the object points.

<u>Solution:</u> All that needs to be done is verify that the image of x under a projective transformation results in the same equation as that stated in the question. The projection of these points gives

$$x_i' = \frac{ax_i + b}{cx_i + d}, \text{ where } ad - bc \neq 0.$$

for $i \in 1, 2, 3, 4$. By direct calculation, we can see that

$$\frac{\left(\frac{ax_1+b}{cx_1+d} - \frac{ax_2+b}{cx_2+d}\right)\left(\frac{ax_3+b}{cx_3+d} - \frac{ax_4+b}{cx_4+d}\right)}{\left(\frac{ax_1+b}{cx_1+d} - \frac{ax_4+b}{cx_4+d}\right)\left(\frac{ax_3+b}{cx_3+d} - \frac{ax_2+b}{cx_2+d}\right)}$$

$$= \frac{((cx_2 + d)(ax_1 + b) - (cx_1 + d)(ax_2 + b))((cx_4 + d)(ax_3 + b) - (cx_3 + d)(ax_2 + b))}{((cx_4 + d)(ax_1 + b) - (cx_1 + d)(ax_4 + b))((cx_2 + d)(ax_3 + b) - (cx_3 + d)(ax_2 + b))}$$

after some tedious algebra. After a little more computing, you can simplify this statement to

$$\frac{((ad - bc)(x_1 - x_2))((ad - bc)(x_3 - x_4))}{((ad - bc)(x_1 - x_4))((ad - bc)(x_3 - x_2))}$$

$$= \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}$$

$$= \frac{(x_1 - x_2)/(x_3 - x_2)}{(x_1 - x_4)/(x_3 - x_4)}$$

which is the same as the cross ratio between object points. Thus, the cross ratio of a line is an invariant under projective transformations of that line.

1-2 Groups. Definition and Examples.

Problem 1-2.1: Show that the cyclic group of order 4 and the Klein 4-group are the only possible structures for groups of order 4.

Solution: This problem would be a very quick exercise if at this point the reader was familiar with Lagrange's theorem, but that hasn't been covered in the text at this point so I will assume the reader doesn't know this one. It's still solvable, but it takes more effort (and paper).

Groups need to fulfill certain conditions, which may be reflected in a Cayley table. In particular, each element in a column (and likewise, a row) may only appear once per column and per row. Further, each element of the group must appear in each column and each row. These greatly reduce the number of binary structures that could represent groups from 4^{16} down to 4. We can write those here:

A	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

B	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

C	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

D	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Now, it turns out that the the tables A, B, C are identical up to an isomorphism. They are

$$\phi_1(A) = \begin{cases} \phi(e) &= e \\ \phi(a) &= b \\ \phi(b) &= a \\ \phi(c) &= c \end{cases} \qquad \phi_2(A) = \begin{cases} \phi(e) &= e \\ \phi(a) &= c \\ \phi(b) &= b \\ \phi(c) &= a \end{cases}$$

which take $A \to B$ and $A \to C$, respectively. We recognize these three as one group, which is the cyclic 4-group. Existence and uniqueness of this group

for any positive integer n is clear from modular arithmetic over the naturals. D is the Klein 4-Group.

Problem 1-2.2: Show that the group of order 4 is Abelian.

Solution: After those two, this one is very quick. We simply observe that the Cayley tables for the order 4 groups are all symmetrical. Thus, the groups they represent are Abelian.

Problem 1-2.3: Give a realization of each group of order 4.

Solution: An example of the cyclic 4-group is $\langle \mathbb{Z}_4, +_4 \rangle$. An example of the Klein 4-group is $\langle \mathbb{Z}_2 \times \mathbb{Z}_2, +_2 \rangle$.

1-3 Subgroups. Cayley's Theorem.

<u>Problem 1-3.1:</u> Give the elements of the regular subgroup of S_6 which is isomorphic with the cyclic group of order 6.

<u>Solution:</u> To see the regular subgroup isomorphic to C_6, we need only write one of the cayley tables for that group and read off the results:

$$
\begin{array}{c|cccccc}
 & e & a & b & c & d & f \\
\hline
e & e & a & b & c & d & f \\
a & a & b & c & d & f & e \\
b & b & c & d & f & e & a \\
c & c & d & f & e & a & b \\
d & d & f & e & a & b & c \\
f & f & e & a & b & c & d \\
\end{array}
\quad \rightarrow \quad
\begin{cases}
\binom{123456}{123456} & = e \\
\binom{123456}{234561} & = (123456) \\
\binom{123456}{345612} & = (135)(246) \\
\binom{123456}{456123} & = (14)(25)(36) \\
\binom{123456}{561234} & = (153)(264) \\
\binom{123456}{612345} & = (165432)
\end{cases}.
$$

We can see that this group is indeed regular. These are all the elements, up to some isomorphisms.

<u>Problem 1-3.2:</u> Use Cayley's theorem to find the possible structures of groups of order 6.

<u>Solution:</u> Cayley's theorem states that every group G of order n is isomorphic with a subgroup of the symmetric group S_n. In our case, $n = 6$. From our work above, we have already done much of our task. We can further break down our result: we presume there are no allowed 6-cycles. Then, because the cycles need be of the same length, we can have two 3-cycles and three 2-cycles, plus e. That gives us

$$\{(123), (132), (12), (13), (23), e\},$$

which happens to be the symmetric 3-group, S_3.

Is that it? It is! How do we know there aren't more? Well, suppose we disallow 3-cycles. Then G has only 2-cycles and the identity. The group has to have 6 elements in total, so we need to have 5 2-cycles as well as the identity. But by the theorem, that puts us at trying to find a group isomorphic to a subset of S_{10}, not S_6. It's just too big.

So there are only two ways to structure a group of order 6: the Abelian group C_6 and the non-Abelian group S_3.

1-4 Cosets. Lagrange's Theorem.

Problem 1-4.1: The cyclic permutations of four symbols form a subgroup $\mathcal{H}$ of S_4. Resolve S_4 into left cosets with respect to $\mathcal{H}$. Compare this resolution with one into right cosets.

Solution: Some of the work has been done for us by this point in the text. We need only look to the previous section to see that the cyclic group of order 4 is given by

$$C_4 = \{e, (1234), (13)(24), (1432)\}.$$

To break up S_4 into cosets partitioned by C_4, we need only apply elements of S_4 that are not present in C_4. Doing this successively and checking to make sure none of your needed elements have already been used, you can construct S_4 in a straightforward manner.

I will write the solution I found and give an example for comparison with the right cosets afterwards: we find that

$$S_4 = C_4 + (12)C_4 + (13)C_4 + (14)C_4 + (23)C_4 + (34)C_4.$$

Nice. Let's compare an example of left and right cosets.

$$(12)C_4 = \{(12), (234), (1324), 143)\}$$

is the left coset. Our right coset is

$$C_4(12) = \{(12), (134), (1423), (243)\}.$$

They're not the same! And in fact, they generally will not be the same — that's because the application of permutations may lead to differing and dependent cycles, which do not commute. Those portions partition portions that are equal for both the left and right coset are called "normal subgroups", and will be covered later in the text. In generality the left and right cosets will have the same index and produce the same results in summation.

Problem 1-4.2: Find the possible structures of groups of order 8.

Solution: By Lagrange's theorem, we know that the elements of the group must have order dividing the order of the group. For a group of order 8, that means the elements may take orders 1,2,4, and 8.

If the group has an order-8 element, it has to be the cyclic group of order 8, C_8.

Next, we treat the case in which there are neither order-8 or order-4 elements. In this case, every element is order-2. Put in another way, this means each element is its own inverse. We shall take a moment here to prove an important property of groups with such a structure of any numver of elements:

Theorem: Any group G where the order of each nontrivial element $a \in G$ is 2 must be abelian.

Proof. suppose $a, b \in G$. Then

$$a \cdot b = (b \cdot b) \cdot a \cdot b \cdot (a \cdot a).$$

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By group axioms, we know that the operation "·" is associative, and we know that the group is closed — therefore, $a \cdot b$ is also an order-2 element by the supposition of this theorem. With that in mind, we see

$$\begin{aligned} a \cdot b &= (b \cdot b) \cdot a \cdot b \cdot (a \cdot a) \\ &= b \cdot (b \cdot a) \cdot (b \cdot a) \cdot a \\ &= b \cdot a. \end{aligned}$$

Ergo, G is an abelian group. Now, for unique elements $a, b \in G$ we can add a third unique element so that we achieve an order-8 group. We call this element c and write

$$G = \{e, a, b, c, ab, ac, bc, abc\}.$$

Even better, we may write that the group is generated in the following way:

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = e, ab = ba, ac = ca, bc = cb \rangle.$$

From here, we can see that this group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, under the transformation $f : G \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where

$$f(a) = (1, 0, 0), \ f(b) = (0, 1, 0), \ f(c) = (0, 0, 1)$$

under combination rules of addition modulo 2.

Now lets suppose that G does in fact have an element of order 4, which we will call x. Then we have a subgroup $H \subset G = \{e, x, x^2, x^3\}$. We can decompose G into cosets, say then that y generates that coset:

$$G = H + yH,$$
$$yH = \{y, yx, yx^2, yx^3\}.$$

Great. Now what kind of elements are in the coset generated by y? Well, let's assume that all the elements in yH are of order 4 and see what happens. If that is the case, then the only element of order-2 in G is x^2. However — if we square any element in the coset, because each is order 4, the result must be order 2. As x^2 is the only order-2 element in G, any squared element of the coset yH must be equal to x^2. We're coming up on a problem:

$$(yx)^2 = y^2 x^2 = x^2 \rightarrow y^2 = e.$$

There's no way for every element of yH to be order-4. We need at least one element of order-2, which we will call y'. We can use that element to generate our coset instead, and so

$$G = H + y'H = \{e, x, x^2, x^3, y', y'x, y'x^2, y'x^3\}.$$

We can see that this is generated by a relation $G = \{y'^i x^j\}$, $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_4$, so G in this case is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

That's it for the abelian cases. We cannot have all elements of order-1 or order-2, and we cannot have an element of order-8. So our remaining cases must contain order-4 elements. Let x be that order-4 element, which again generates a subgroup $H \subset G = \{e, x, x^2, x^3\}$. This subgroup of G gives an index of 2, so H has two left (or right) cosets. Additionally, this subgroup itself is abelian. Therefore,

$$\text{if } g \in H, gH = H = Hg.$$

But because H has partitioned G into just two cosets, if

$$g \notin H \rightarrow gH = G/H \rightarrow Hg = G/H$$

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which means that H is a normal subgroup. That hasn't come up in the text as of yet, so we're going to pretend that we don't know all about those by now and discover the property we're going to need: namely, we're just going to observe that for groups that have the property $gH = Hg$, we may write $H = gHg^{-1}$. We call this flanking by the coset generator and its inverse "conjugation". We need a further result that is also not yet in the text:

Theorem: The operation of conjugation preserves the order of the conjugated element.

Proof. Suppose $b = gag^{-1}$. Then $b^k = (gag^{-1})^k$. Then

$$b^k = (gag^{-1})^k = gag^{-1}gag^{-1}\ldots gag^{-1} = g(aea\ldots eaea)g^{-1} = ga^k g^{-1}.$$

$\blacksquare$

As a corollary, if b is order m then by the theorem above, $e = b^m = ga^m g^{-1} \to a^m = e$ as those are the only order-4 elements. In the case where $gxg^{-1} = x$, that's an abelian group again and we've already got that accounted for.

So if $gxg^{=1} = x^3$, that's the same as saying $gxg^{=1} = x^{-1}$. In the case where g is of order-2, our group G is generated by the following relations:

$$gxg^{-1} = x^{-1}$$
$$x^4 = e$$
$$g^2 = e$$

which are the defining properties of the dihedral group of order-8, D_8. These are the symmetries of a square, 2 order-4 operations (corresponding to rotation clockwise and counter-clockwise by $\frac{\pi}{4}$), an order-1 operation (rotating either way by 2π), and an order-2 operation (covering rotation of $\frac{\pi}{2}$ in either direction). We have also 4 other order-2 operations, 2 reflections through the corners, and 2 through the sides.

Our last case is the same as the above but we consider all elements in G/H to be order-4. Then say we take $g \in G/H$: we can see that $g^2 = x^2$ becuase g^2 is order-2 and the only order-2 element in G is x^2. We also see that G is non-abelian. If it were, and $gx = xg$, then

$$(x^3 g)^2 = x^3 g x^3 g = x^6 g^2 = x^2 g^2 = x^2 x^2 = x^4 = e,$$

contradicting the fact that the element x^3g is order-4. Further, we cannot have $gx = x^2g$ because

$$gx = x^2g = g^2g = g^3 \rightarrow x = g^2.$$

x is order-4 and g^2 is order-2, so that's just impossible. So we are left with the realization that $gx = x^3g$. That's enough to give us a Cayley table, so we may write

$$G = \langle x, g \mid x^4 = g^4 = e, x^2 = g^2, gx = x^3g \rangle.$$

This is the quaternion group, and it is fundamental to particle physics.

1-5 Conjugate Classes.

Problem 1-5.1: Seperate the elements of S_5 into conjugate classes.

Solution: The conjugate classes of S_5 are as follows:

1. We have the identity cycle, the collection of 1-cycles on 5 elements. This has just a single element, e.

2. We have the single 2-cycles, the collection of distinct cycles over just two elements. This has $\binom{5}{2} = 10$ elements.

3. We have the collections of independent double 2-cycles, two 2-cycles over independent elements: This class has $\binom{5}{2} \cdot \binom{3}{2} \cdot \frac{1}{2} = 15$ elements, where

4. We have the class of single 3-cycles, which have $\binom{5}{3} \cdot 2! = 20$ elements (the factorial from the number of internal arrangements allowable within a cycle over 3 elements — $(123) \neq (132)$, etc.

5. We have the collection of independent 3-cycle and 2-cycle arrangements, which have $\binom{5}{3} \cdot 2! = 20$ elements. This follows exactly the same logic as the previous case, as we have no choice in our 2-cycle once the 3-cycle is determined.

6. We have the class of 4-cycles, which have $\binom{5}{4} \cdot 3! = 30$ elements.

7. We have the class of 5-cycles, which simply have $4! = 24$ elements.

If we add the number of elements up, we can see that we get $120 = 5!$, which is exactly what we should expcect from the symmetric group over 5.

Problem 1-5.2: Continue the table to $n = 5, 6, 7$. For $n = 5$, give the cycle structure corresponding to each partition.

Solution:

$$S_5 : (5), (41), (32), (31^2), (2^21), (21^3)(1^5)$$
$$S_6 : (6), (51), (42), (41^2), (3^2), (321), (31^3), (2^3), (2^21^2), (21^4), (1^6)$$
$$S_7 : (7), (61), (52), (51^2), (43), (421), (41^3), (3^21), (32^2), (321^2), (31^4),$$
$$(2^31), (2^21^3), (21^5), (1^7)$$

For S_5, the cyclic structures are given by

$$(abcde), (abcd), (abc)(de), (abc), (ab)(cd), (ab) \text{ and } e,$$

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respectively.

Problem 1-5.3: Find the number of permutations in each class of S_5.

Solution: Already done, see the solution for 1-5.1. Got ahead of myself! However, I will use the formula for one case to illustrate how it is done: Say for the partition (32) in the table, we have $n = 5$ and so

$$\frac{n!}{\prod_i^n \nu_i! i^{\nu_i}} = \frac{120}{(0!(1^0)(4^0)(5^0))(1!2^1)(1!3^1)} = \frac{120}{6} = 20,$$

as there is one 3-cycle and one 2-cycle in this partition, and zero 1-, 4-, and 5-cycles.

Problem 1-5.4: By taking successive products of the permutations

$$(1234)(5678) \text{ and } (1537)(2846)$$

show that one generates a group of order 8. Seperate the elements of the group into conjugate classes. Show that this group (the quaternion group) is isomorphic to the group with elements

$$1, -1, i, -i, j, -j, k, -k; i^2 = j^2 = k^2 = -1, \ ij = k, jk = i, ki = j.$$

Solution: Let $a = (1234)(5678)$ and $b = (1537)(2846)$. Then from our previous work discovering the quarternion group, we can see that the group elements are going to be without doing much work — though it is straightforward to verify that this is the group:

$$G = \langle e, a, a^2, a^3, b, ab, b^3, ba \rangle.$$

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We see a few important relations that will let us define the isomorphism. Those relations are:

1. $a^4 = b^4 = e$,

2. $a^2 = b^2$,

3. $a^3 = ab^2 = b^2 a$,

4. $b^3 = ba^2 = a^2 b$,

5. $ba = a^3b$.

So seeing a^2 is its own element and $(a^2)^2 = 1$, with 1 already taken, we assign $a^2 = -1$. Further, we assign $a = i$. Then $a^3 = -i$. We assign $b = j$, $b^3 = -j$, and $ab = k$, $ba = -k$. This assignment satisfies all the requirements of the quaternion group and follows the same group relations.

1-6 Invariant subgroups. Factor groups. Homomorphism.

<u>Problem 1-6.1:</u> Without enumerating, find the conjugate subgroups of S_3 in S_4; of S_2 in S_4; of the cyclic group $\{e, (123), 132)\}$ in S_4.

<u>Solution:</u> It is apparent that what this admittedly older text calls "invariant subgroups" are what are now usually termed "normal subgroups", and that what Hamermesh calls "factor groups" are what others would probably call "quotient groups" today. Anyway, we carry on: I'm not certain what is meant by enumerating in this context — if they mean not to use combinatorial tricks to find the number of conjugate subgroups and their orders, then not to worry. For S_3 in S_4, these are the subgroups generated by permutations that fix each of the indiviual elements in $\{1, 2, 3, 4\}$.

$$H = \{(), (12), (13), (23), (123), (132)\} = H_4,$$
$$H_1 = \{().(23), (34), (24), (234), (243)\},$$
$$H_2 = \{().(13), (14), (34), (134), (143)\},$$
$$H_3 = \{().(12), (14), (24), (124), (142)\}.$$

For S_2 in S_4, this is going to be the subgroups generated by the permutations that fix 2-cylces of elements of $\{1, 2, 3, 4\}$. The conjugate subgroups, then, are

$$H = \{(), (12)\} = H_{34},$$
$$H_{24} = \{(), (13)\},$$
$$H_{23} = \{(), (14)\},$$
$$H_{14} = \{(), (23)\},$$
$$H_{13} = \{(), (24)\},$$
$$H_{12} = \{(), (34)\}.$$

And finally, for the conjugate subgroup generated by $\{(), (123), (132)\}$, we

have

$$H_4 = \{(), (123).(132)\},$$
$$H_3 = \{(), (124).(142)\},$$
$$H_2 = \{(), (134).(143)\},$$
$$H_1 = \{(), (234).(243)\}.$$

Problem 1-6.2: Prove that a subgroup of index 2 is necessarily invariant.

Solution: See 1-4.2.

Problem 1-6.3: Show that all subgroups of the quarternion group are invariant.

Solution: The element 1 commutes with each element, it's the identity. So the trivial subgroup is invariant. The arithmetic is also straightforward for the elements $-1, i, j, k$ individually.

Each of i, j, k generate order-4 subgroups which in this case makes them index-2. By the theorem proved in 1-4.2, that makes those subgroups necessarily invariant. -1 itself generates an order-2 subgroup, which is in fact normal as -1 commutes with each element.

These are the only proper subgroups (by Lagrange's theorem), and so each subgroup of the quaternions is invariant, or normal.

CHAPTER 2: SYMMETRY GROUPS

2-3 Groups whose elements are pure rotations: uniaxial groups, dihedral groups.